TRINITY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

YEAR 12 2010 ASSESSMENT TASK 3

MATHEMATICS EXTENSION 2

HALF YEARLY EXAMINATION

Time Allowed – *3 hours plus 5 minutes reading*

WEIGHTING 30% towards final assessment

Outcomes referred to: E2, E3, E6, E7, E8 and E9

Date Tuesday 27th April Morning Session

INSTRUCTIONS:

1. Attempt **ALL** questions.
2. Show all necessary working.
3. **Begin** each question in a **new booklet**.
4. Mark values are shown beside each part. Total marks 120.
5. Non-programmable silent Board of Studies approved calculators are permitted.
6. Place your teacher's name on each booklet.
7. If requested, additional writing booklets may be obtained from a supervisor on request.
8. A table of standard integrals is provided at the back of this paper.
9. All necessary working should be shown in every question.

Question 1. Use a SEPARATE page. **Marks**

(a) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are not asked to find the primitive functions)

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 \theta \, d\theta = 0$ **2**

(ii) $\int_{0}^{\pi} \sin^7 \theta \, d\theta = 0$ **2**

(iii) $\int_{-1}^{1} e^{-x^2} \, dx = 0$ **2**

(iv) $\int_{0}^{\frac{\pi}{2}} \left(\sin^8 \theta - \cos^8 \theta\right) d\theta = 0$ **2**

(b) Find the exact value of :

(i) $\int_{0}^{1} \frac{2x}{1+2x} \, dx$ **2**

(ii) $\int_{1}^{e} \frac{\left(\log_e x\right)^2}{x} dx$ **2**

(iii) $\int_{0}^{\frac{1}{2}} \cos^{-1} x \, dx = 0$ **3**

Question 2. Use a SEPARATE page. **Marks**

(a) Indicate on an Argand diagram the region in which z lies given both $|z-(3+i)|\leq 3$ and $\frac{\pi}{4}<\arg[z-(1+i)]\leq\frac{\pi}{2}$ are satisfied. 3

(b) Find the locus in the Argand diagram of the point P which represents the complex number z where $z\bar{z}-4\left(z+\bar{z}\right)=9$. 3

(c) Show by geometrical considerations or otherwise that if complex numbers z_1 and z_2 are such that $|z_1|=|z_2|$ then $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary. 2

(d) Sketch the circle C with Cartesian equation $x^2+(y-1)^2=1$. The point P, representing the non zero complex number z, lies on C. 1

(i) Express Z in terms of θ, the argument of z. 2

(ii) Given that $z'=\frac{1}{z}$, find the modulus and argument of z' in terms of θ. 2

Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line and determine the equation of this line. 2

Question 3. Use a SEPARATE page. **Marks**

(a) Find: $\int cosec\ x\ \ dx$ by using the substitution $t = \tan\frac{x}{2}$. **2**

(b) Find: $\int \frac{dx}{x\left(1+x^2\right)}$. **2**

(c) Find: $\int \frac{dx}{x\sqrt{x^2-1}}$. (Hint: use substitution $u = \sqrt{x^2-1}$). **2**

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\, dx$. (Hint: use t results). **4**

(e) Given that $I_n = \int sec^n\ x\ \ dx$, where $n \geq 2$, show that
$(n-1)\,I_n = \tan x\ \sec^{n-2} x + (n-2)I_{n-2}$. **5**
Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x\ \ dx$.

Question 4. Use a SEPARATE page. **Marks**

(a) Given that $z_1 = 3 - i,\ z_2 = 2 + 5i,$ express in the form of a +ib, where a, b are real,

(i) $\left(\overline{z_1}\right)^2$ 2

(ii) $\dfrac{z_1}{z_2}$ 2

(iii) $\left|\dfrac{\overline{z_1}}{z_2}\right|.$ 2

(b) Given that for the complex number z, $|z| = 2, \quad \arg z = \dfrac{2\pi}{5},$
write in the form of a +ib, where a, b are real,

(i) z. 2

(ii) z^7. 2

(c) Draw a sketch of the region of the Argand diagram consisting of the set of all values of z for which $1 \leq |z_1| \leq 4$ and $\dfrac{\pi}{4} < \arg z \leq \dfrac{3\pi}{4}$. 2

(d) Reduce the complex expression $\dfrac{(2-i)(8+3i)}{(3+i)}$ to the form of a +ib, where a, b are real numbers. 3

Question 5. Use a SEPARATE page. **Marks**

(a) The function *f(x)* is given by $f(x)=\frac{4(2x-7)}{(x-3)(x+1)}$.

(i) Express *f(x)* in partial fractions. 2

(ii) Sketch the graph $y = f(x)$ showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points and the equations of any asymptotes. (There is no need to investigate points of inflexion). 2

(iii) Determine the area of the region bounded by the curve, the x-axis and the lines $x = 4$ and $x = 6$, expressing your answer as a single logarithm. 2

(b) Let $f(x)=-x^2+6x-8$. On separate diagrams, using half a page for each, and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y=f(x)$. 1

(ii) $y=|f(x)|$. 1

(iii) $y^2=f(x)$. 1

(iv) $y=\frac{1}{f(x)}$. 1

(v) $y=e^{f(x)}$. 1

(c) (i) Obtain the equation of the tangent to the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ at the point $P\left(x_1, y_1\right)$ on the curve. 2

(ii) This tangent meets the coordinate axes at Q and R. Show that $OQ+OR=a$ for all positions of P, where O is the origin. 2

Question 6. Use a SEPARATE page. **Marks**

(a) The numbers a, b, c are said to be in harmonic progression if their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and b is then said to be the harmonic mean of a and c.

(i) Show that the numbers 6, 8, 12 are in harmonic progression. **4**

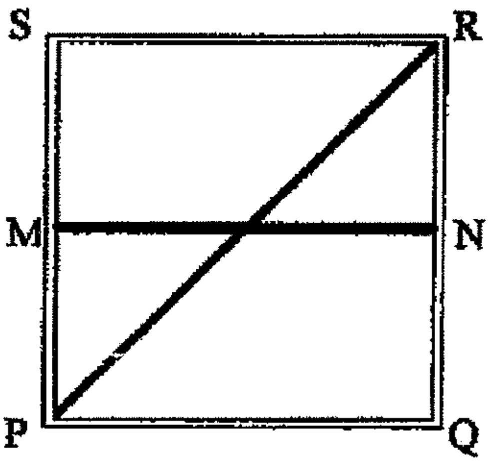
(ii) Show that harmonic mean of a and c is equal to $\frac{2ac}{a+c}$. **4**

(b) (i) Show that $\tan\left(A+\frac{\pi}{2}\right)=-\cot A$ **3**

(ii) Use the method of mathematical induction to show that $\tan\left((2n+1)\frac{\pi}{4}\right)=(-1)^n$ for all integers $n \geq 1$. **4**

Question 7. Use a SEPARATE page. **Marks**

(a) The Anti-Bureaucracy Organisation wants to install a rectangular noticeboard PQRS of fixed area A square metres in each of its offices. The noticeboard is to be subdivided by two thin strips of red tape PR and MN (where MN is parallel to PQ) as shown below.

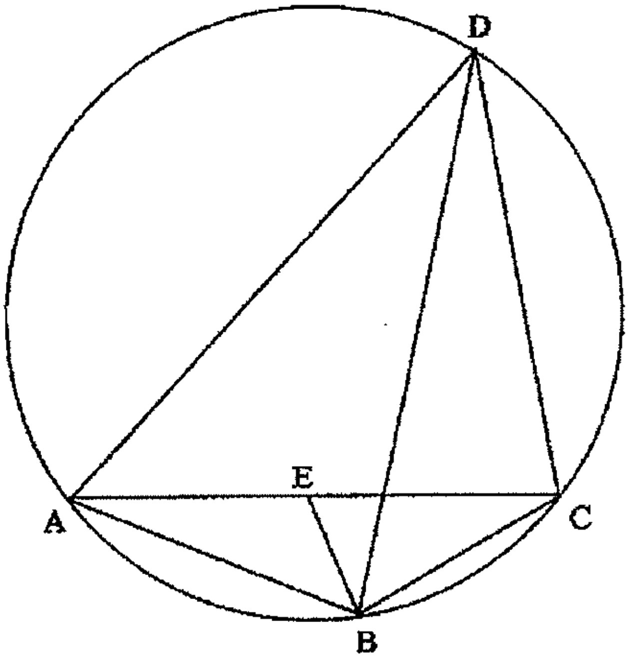


Find, in terms of A, the dimensions of the notice board so that the length of red tape used is a minimum. 7

(b) (i) Sketch the curve $9y^2 = x(x-3)^2$ showing clearly the coordinates of any turning points. 4

(ii) Show that the area enclosed between the x axis and that part of the curve which lies in the first quadrant between $x = 0$ and $x = 3$ is $\frac{4\sqrt{3}}{5}$ square units. 4

Question 8. Use a SEPARATE page. **Marks**

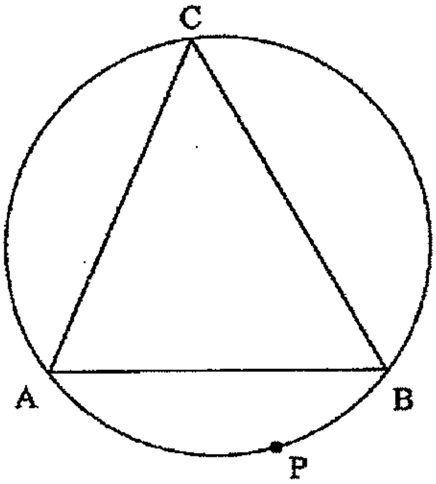


(a) In the diagram ABCD is a cyclic quadrilateral. E is the point on AC such that $\angle ABE = \angle DBC$. Copy the diagram into your work.

(i) Show that $\Delta ABE /// \Delta DBC$ and $\Delta ABD /// \Delta EBC$ 6

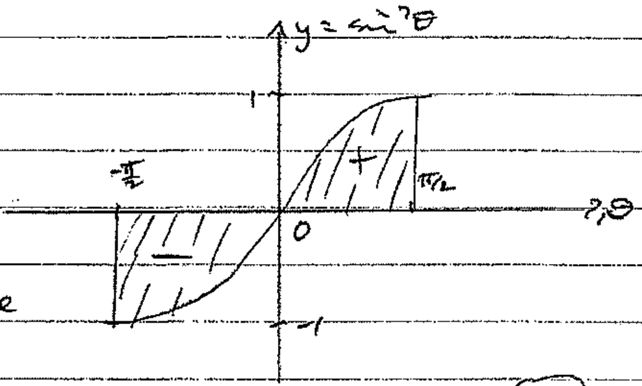
(ii) Hence show that $(AB)(DC) + (AD)(BC) = (AC)(DB)$ 4

(b) In the diagram ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. Use the result above to show that PC = PA + PB. 5



2010 EXT 2 APRIL EXAM SOLUTIONS

1 (a) (i) $\int_{-\frac{\pi}{2}}^{\pi/2} \sin^7\theta . d\theta = 0$

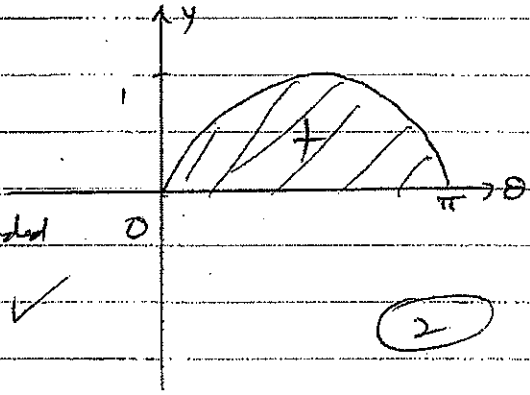


✓

True as $\int_{-\frac{\pi}{2}}^{0} y . d\theta = -\int_{0}^{\pi/2} y d\theta$ due to y being an odd function. ✓

②

(ii) $\int_{0}^{\pi} \sin^7\theta . d\theta = 0$

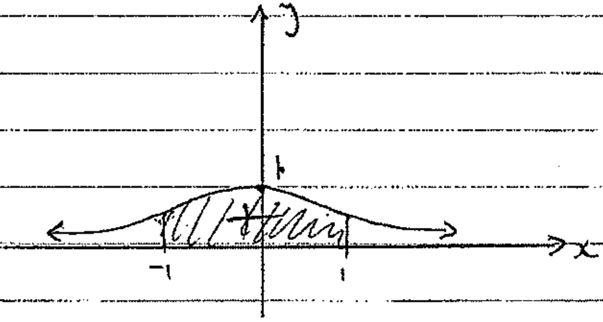


✓

False as area is equivalent to shaded area shown ie greater than 0. ✓

②

(iii) $\int_{-1}^{1} e^{-x^2} . dx = 0$



False as area is equivalent to shaded area shown ie greater than 0 or even fn.

(iv) $\int_{0}^{\frac{\pi}{2}} (\sin^8\theta - \cos^8\theta) . d\theta = 0$

$= \int_{0}^{\pi/2} \sin^8\theta . d\theta - \int_{0}^{\frac{\pi}{2}} \cos^8\theta d\theta$

②

True ✓ as $\int_{0}^{\pi/2} \sin^8\theta d\theta = \int_{0}^{\pi/2} \cos^8\theta d\theta$ ✓ due to $\sin\theta = \cos(\frac{\pi}{2} - \theta)$

(b) (i) $\int_0^1 \frac{2x}{1+2x}\,dx = \int_0^1 \left(1 - \frac{1}{1+2x}\right)dx$ $\quad \begin{array}{r} 1 \\ 1+2x\overline{)\,2x} \\ 2x+1 \\ -1 \end{array}$

$= \left[x - \frac{1}{2}\ln(1+2x)\right]_0^1$ ✓

$= 1 - \frac{1}{2}\ln 3 - (0-0)$

②

$= 1 - \frac{1}{2}\ln 3$ ✓

(ii) $\int_1^e \frac{(\log_e x)^2}{x}\,dx = \left[\frac{(\log_e x)^3}{3}\right]_1^e$

$= \frac{1^3}{3} - \frac{0^3}{3}$ ②

$= \frac{1}{3}$ ✓

1(b) (iii) $\int_0^{1/2} \cos^{-1} x \,.\, dx = \int_0^{1/2} 1 \,.\, \cos^{-1} x \,.\, dx$

$= \left[x \,.\, \cos^{-1} x\right]_0^{1/2} - \int_0^{1/2} x \,.\, \frac{-1}{\sqrt{(1-x^2)}} \,.\, dx$ ✓

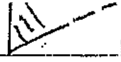
$= \left[x \cos^{-1} x\right]_0^{1/2} + \left(\frac{-1}{2}\right)\int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} \,.\, (-2x) \,.\, dx$

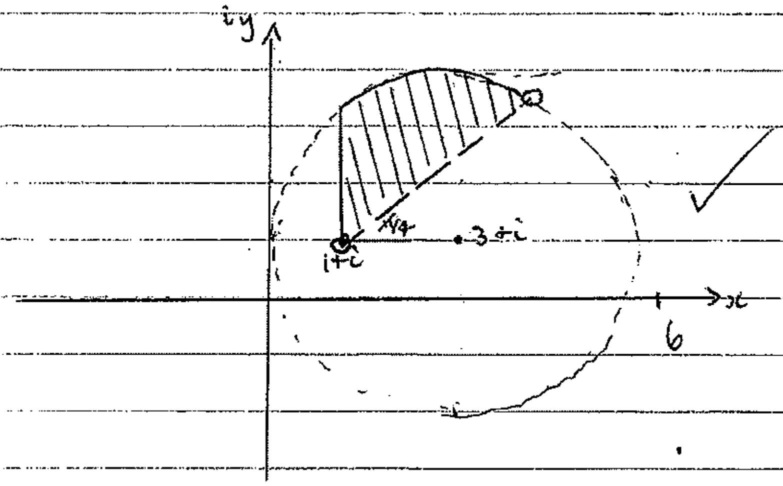
$= \left[x \cos^{-1} x - (1-x^2)^{\frac{1}{2}}\right]_0^{1/2}$

$= \left(\frac{1}{2} \,.\, \frac{\pi}{3} - \sqrt{\frac{3}{4}}\right) - \left(0 - 1\right)$

$= \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$ ✓ ③

2 (a) $|z-(3+i)| \le 3 \Rightarrow$ Inside circle centre (3,1) radius 3. ✓

$\frac{\pi}{4} < \arg[z-(1+i)] \le \frac{\pi}{2} \Rightarrow$ V centre 1+i  ✓



✓ ③

(b) $z.\bar{z} - 4(z+\bar{z}) = 9$, putting $z = x+iy$

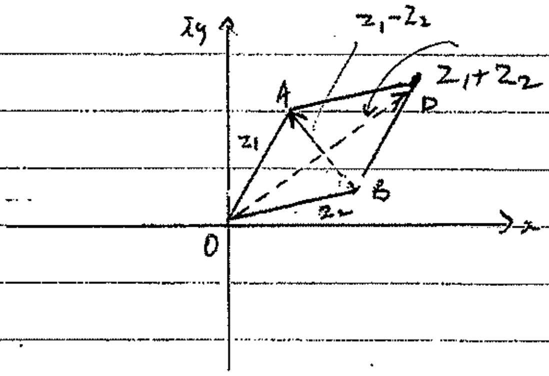
$(x+iy)(x-iy) - 4(x+iy+x-iy) = 9$ ✓ ③

$x^2+y^2 - 8x = 9$

$x^2-8x+16 + y^2 = 25$

$(x-4)^2 + y^2 = 25$ ✓ $\Rightarrow$ locus is a circle centre (4,0) rad 5. ✓

2 (c) $|z_1| = |z_2|$ ⇒ same modulus.

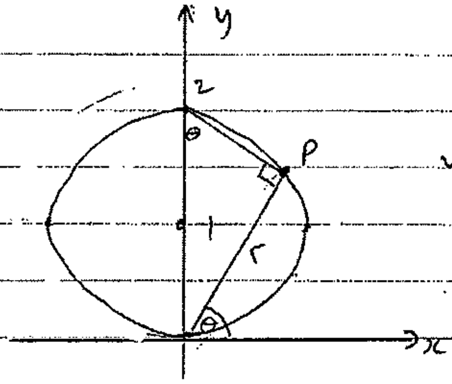


As $|z_1| = |z_2|$ then //ogram OADB is a rhombus ⇒ diagonals AB ⊥ OD. ✓

∴ $z_1 + z_2 = ki(z_1 - z_2)$ for some $k \in R$.

∴ $\frac{z_1+z_2}{z_1-z_2} = ki$ ✓ is purely imaginary. ②

(d)



✓ ①

(i) At $\theta = 0$ $|z| = 0$ | $\sin\theta = \frac{r}{2}$

$\theta = \frac{\pi}{2}$ $|z| = 2$ | ∴ $r = 2\sin\theta$

$\theta = \frac{\pi}{4}$ $|z| = \sqrt{2}$

✓ ✓

⇒ $|z| = 2\sin\theta$ ②

(ii) $z' = \frac{1}{z}$ ∴ $|z'| = \left|\frac{1}{z}\right| = \frac{1}{2\sin\theta}$

$= \frac{1}{2}\operatorname{cosec}\theta$. ✓ ②

$\arg z' = \arg\left(\frac{1}{z}\right) = -\theta$ ✓

2(d) con'd

$$p' = \tfrac{1}{2}\,\text{cosec}\,\theta\left(\cos(-\theta) + i\sin(-\theta)\right)$$

$$= \tfrac{1}{2}\,\text{cosec}\,\theta\left(\cos\theta - i\sin\theta\right)$$

$$= \tfrac{1}{2}\cot\theta - \tfrac{1}{2}i\,\frac{\sin\theta}{\sin\theta}$$

$$= \tfrac{1}{2}\cot\theta - \tfrac{1}{2}i \quad . \quad ✓$$

∴ p' has $y = -\frac{1}{2}$ (irrespective of θ) which is a st. line. ✓ ②

3(a) $\int \text{cosec}\,x \, . \, dx = \int \frac{1}{\sin x} \, . \, dx$

$$\left.\begin{aligned}\frac{dt}{dx} &= \tfrac{1}{2}\sec^2\frac{x}{2}\\ &= \tfrac{1}{2}(1+t^2)\end{aligned}\right] \; ✓ \quad \Rightarrow \quad = \int \frac{1+t^2}{2t}\,.\,\frac{1}{\frac{1}{2}(1+t^2)}\,.\,dt$$

$$= \int \frac{1}{t}\,dt$$

$$= \ln|t| + c \quad ②$$

$$= \ln\left|\tan\left(\tfrac{x}{2}\right)\right| + c \quad ✓$$

(b) $\int \frac{dx}{x(1+x^2)} = \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$ ✓

$$= \ln|x| - \tfrac{1}{2}\ln|(1+x^2)| + c$$

$$= \ln\left|\left(\frac{x}{\sqrt{1+x^2}}\right)\right| + c \quad ✓ ②$$

3 con'd (c) Put $u = \sqrt{x^2-1}$ $\Rightarrow$ $u^2+1 = x^2$

$$\frac{du}{dx} = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}.2x$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{du}{u^2+1}$$ ✓

$$= \tan^{-1} u + c$$ ②

$$= \tan^{-1}\left(\sqrt{x^2-1}\right) + c.$$ ✓

(d) $$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\cdot dx = \int_0^1 \frac{\frac{2}{1+t^2}\cdot dt}{3+5\left(\frac{1-t^2}{1+t^2}\right)}$$ ✓ (ignore limits for this mark.)

$$= \int_0^1 \frac{2\,dt}{3(1+t^2)+5(1-t^2)}$$

$$= \int_0^1 \frac{2\,dt}{8-2t^2}$$

$$= \int_0^1 \frac{dt}{4-t^2}$$ ✓

$$= \int_0^1 \frac{1}{4}\left(\frac{1}{2-t}+\frac{1}{2+t}\right)dt$$

$$= \left[\frac{1}{4}\ln\frac{(2+t)}{(2-t)}\right]_0^1$$ ✓

$$= \frac{1}{4}\ln\frac{(2+1)}{(2-1)} - \ln\frac{2}{2}$$

$$= \frac{1}{4}\ln 3.$$ ✓ ④

(e) $I_n = \int \sec^n x . dx$

$= \int \sec^2 x . \sec^{n-2} x . dx$

$= \tan x . \sec^{n-2} x - \int \tan^2 x . (n-2) . \sec^{n-3} x . \sec x . dx$ ✓

$= \tan x . \sec^{n-2} x - (n-2) \int (\sec^2 x - 1) . \sec^{n-2} x \, dx$

$= \tan x . \sec^{n-2} x - (n-2) \int \sec^n x . dx + (n-2) \int \sec^{n-2} x . dx$ ✓

$\therefore I_n = \tan x . \sec^{n-2} x - (n-2) . I_n + (n-2) I_{n-2}$

$\therefore I_n + (n-2) I_n = \tan x . \sec^{n-2} x + (n-2) I_{n-2}$

ie $(n-1) I_n = \tan x . \sec^{n-2} x + (n-2) I_{n-2}$ ✓

Hence $I_n = \frac{1}{n-1} \tan x . \sec^{n-2} x + \frac{(n-2) I_{n-2}}{(n-1)}$

$\therefore \int_0^{\pi/4} \sec^6 x . dx = \left[\frac{1}{6-1} . \tan x . \sec^4 x\right]_0^{\frac{\pi}{4}} + \frac{6-2}{6-1} . \int_0^{\pi/4} \sec^4 . dx$ ✓

$= \frac{1}{5} . 1 . (\sqrt{2})^4 + \frac{4}{5}\left[\frac{1}{3} . 1 . (\sqrt{2})^2 + \frac{2}{3}\left(\frac{1}{1} - 0\right)\right]$

$= \frac{4}{5} + \frac{4}{5} . \frac{4}{3}$

$= \frac{28}{15}$ ✓ ⑤

4 (a) $z_1 = 3 - i \ , \ z_2 = 2 + 5i$

(i) $(\bar{z}_1)^2 = (3+i)^2$ ✓

$= 9 + 6i - 1$

$= 8 + 6i$ ✓ ②

(ii) $\frac{z_1}{z_2} = \frac{3-i}{2+5i} \times \frac{2-5i}{2-5i}$ ✓

$= \frac{6 - 15i - 2i - 5}{4 + 25}$

$= \frac{1}{29} - \frac{17}{29} i$ ✓ ②

(iii) $\left| \frac{\bar{z}_1}{z_2} \right| = \frac{|\bar{z}_1|}{|z_2|}$ ✓

$= \frac{\sqrt{9+1}}{\sqrt{29}}$

$= \sqrt{\frac{10}{29}}$ ✓ ②.

(b) $|z| = 2 \ , \ \arg z = \frac{2\pi}{5}$

(i) $z = \overset{✓}{2}\left(\cos\frac{2\pi}{5} + \overset{✓}{i} \sin\frac{2\pi}{5} \right)$ (1 for 2, 1 for cis)

$= 2\cos\frac{2\pi}{5} + i . 2\sin\frac{2\pi}{5}$ ②.

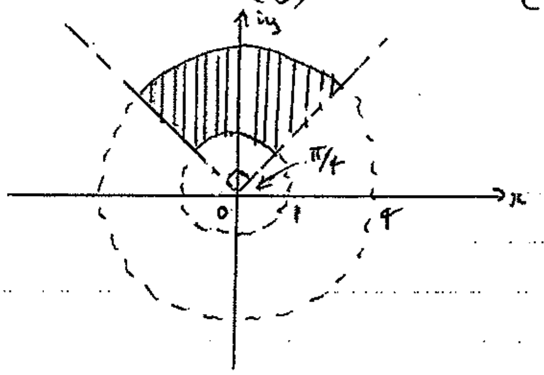
✓

(ii) $z^7 = 2^7\left(\cos \frac{7.2\pi}{5} + i \sin \frac{7.2\pi}{5} \right)$

✓

$= 128 \cos\left(\frac{4\pi}{5}\right) + 128 \sin\left(\frac{4\pi}{5}\right)$ ②.

(c)



②

4 con'd.

(d) $\frac{(2-i)(8+3i)}{3+i} = \frac{16+6i-8i+3}{3+i} \checkmark \times \frac{3-i}{3-i}$

$= \frac{19-2i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{57-19i-6i-2}{9+1}$ ✓

$= \frac{55-25i}{10}$ ③

$= \frac{11}{2} - \frac{5}{2}i$ ✓

5 (a) (i) $\frac{4(2x-7)}{(x-3)(x+1)} = \frac{a}{x-3} + \frac{b}{x+1}$

$= \frac{a(x+1)+b(x+3)}{(x-3)(x+1)}$ ✓

$\therefore \quad a+b = 8$

$a-3b = -28$

$4b = -36$

$b = 9$

$a = -1$

$\therefore \quad f(x) = \frac{9}{x+1} - \frac{1}{x-3}$ ✓ ②

(ii) Asymptotes at $x = 3, -1$.

At $x = 0$, $f(x) = 9\frac{1}{3}$

$y = 0$, $x = 3\frac{1}{2}$

$f'(x) = \frac{-9}{(x+1)^2} + \frac{1}{(x-3)^2} = 0$

At $(x+1)^2 = 9(x-3)^2$

$x^2+2x+1 = 9x^2-54x+81$

$8x^2-56x^2+80 = 0$

$4x^2-28x+40 = 0$

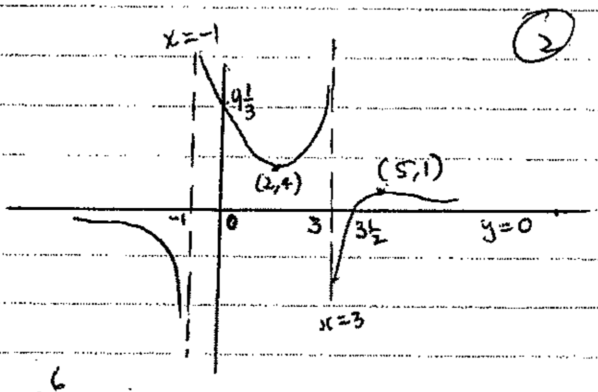
5 con'd.

$$x^2 - 7x + 10 = 0$$

$$(x-5)(x-2) = 0$$

$$\Rightarrow x = 2 \text{ or } 5.$$

$$\therefore \quad y = 4 \text{ or } 1$$



② ✓shape

✓{asymptotes
axes crossed
T.P.'s.

(iii) $$A = \int_4^6 \left(\frac{9}{x+1} - \frac{1}{x-3}\right) dx$$

$$= \left[\ln \frac{(x+1)^9}{(x-3)} \right]_4^6 \quad ✓$$

$$= \ln \frac{7^9}{3} - \ln \frac{5^9}{1}$$

$$= \ln \frac{7^9}{3} \times \frac{1}{5^9}$$ ②

$$= \ln \left(\frac{7^9}{3 \times 5^9} \right) \quad ✓$$

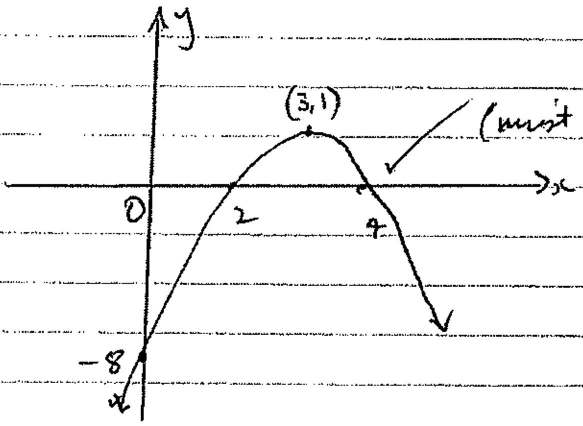
(b) $$f(x) = -x^2 + 6x - 8$$

$$= -(x^2 - 6x + 8)$$

$$= -(x-4)(x-2)$$

$$f(0) = -8$$

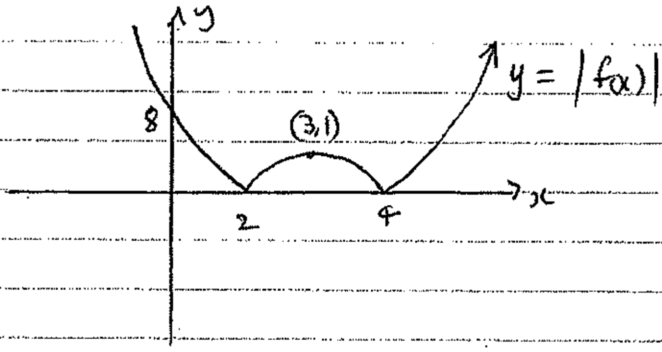
(i) $y = f(x)$



✓ (must show intercepts)

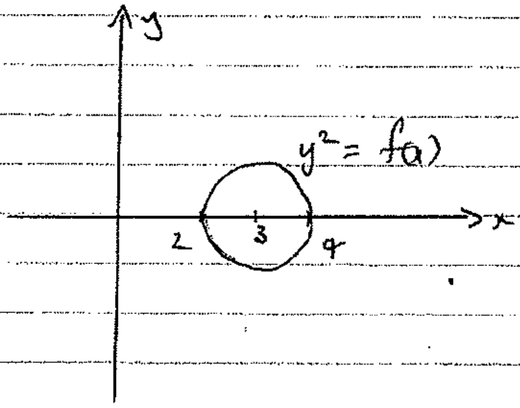
①

5(b) (ii)



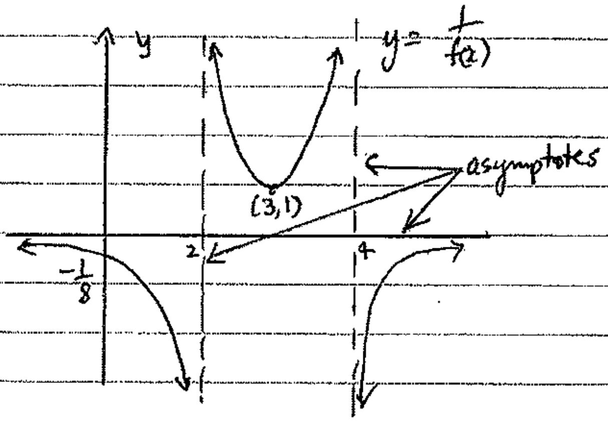
①

(iii)



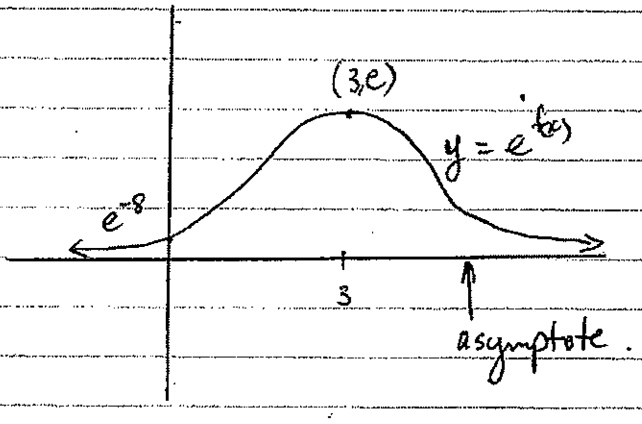
①

(iv)



①

(v)



①

5 c (i) $x^{\frac{1}{2}} + y^{1/2} = a^{1/2}$

$$\tfrac{1}{2} x^{-\frac{1}{2}} + \tfrac{1}{2} y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$$

$$= -\sqrt{\frac{y}{x}} \quad ✓$$

$\therefore$ Through $P(x_1, y_1)$

$$\frac{dy}{dx} = -\sqrt{\frac{y_1}{x_1}}$$

$\therefore$ Eqⁿ of tangent

$$y - y_1 = -\sqrt{\frac{y_1}{x_1}}\,(x - x_1) \quad ✓ \quad ②$$

(ii)

$$\left.\begin{aligned} &\text{If } x = 0, \quad y = y_1 - \sqrt{\frac{y_1}{x_1}}\,(-x_1) \\ &\text{If } y = 0 \quad -y_1 = -\sqrt{\frac{y_1}{x_1}}\,(x - x_1) \\ &\therefore\ y_1\sqrt{\frac{x_1}{y_1}} = x - x_1 \\ &\qquad x = x_1 + y_1\sqrt{\frac{x_1}{y_1}} \end{aligned}\right\}\text{either 1.} \quad ✓$$

$$\therefore\ OQ + OR = y_1 - \sqrt{\frac{y_1}{x_1}}\,(-x_1) + x_1 + y_1\sqrt{\frac{x_1}{y_1}}$$

$$= y_1 + \sqrt{x_1 y_1} + x_1 + \sqrt{x_1 y_1}$$

$$= x_1 + 2\sqrt{x_1 y_1} + y_1$$

$$= \left(x_1^{1/2} + y_1^{1/2}\right)^2 \quad ✓ \quad ②$$

$$= \left(a^{1/2}\right)^2$$

$$= a.$$

6. (a) (i) $\frac{1}{6}, \frac{1}{8}, \frac{1}{12}$

Put $T_1 = \frac{1}{6}$, $T_2 = \frac{1}{8}$, $T_3 = \frac{1}{12}$

$T_2 - T_1 = \frac{1}{8} - \frac{1}{6}$ ✓

$= \frac{-2}{48} = \frac{-1}{24}$ ✓

$T_3 - T_2 = \frac{1}{12} - \frac{1}{8}$

$= \frac{-4}{96}$

$= -\frac{1}{24}$ ✓

④

$\therefore T_2 - T_1 = T_3 - T_2$ ✓

$\therefore \frac{1}{6}, \frac{1}{8}, \frac{1}{12}$ are in arithmetic progression

$\Rightarrow$ 6, 8, 12 ✓ are in harmonic progression

(ii) Arith Mean $\frac{1}{b} = \frac{\frac{1}{a} + \frac{1}{c}}{2}$ ✓

$= \frac{a+c}{2ac}$ ✓

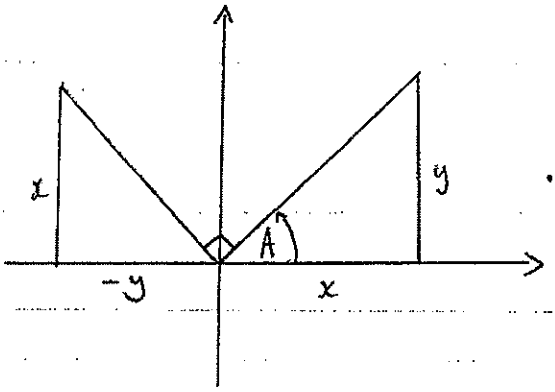
④

$\therefore b = \frac{2ac}{a+c}$ ✓

6 (b) (i) R.T.Show

$$\tan\left(A+\frac{\pi}{2}\right) = -\cot A$$

Put $\frac{y}{x} = \tan A$ (as shown below)



✓

$$\therefore \tan\left(A+\frac{\pi}{2}\right) = \frac{x}{-y}$$ ✓

$$-\cot A = -\left(\frac{x}{y}\right)$$

③

$$\therefore \tan\left(A+\frac{\pi}{2}\right) = -\cot A.$$

If A is taken into 2nd quadrant then there is a change of sign for $\tan\left(A+\frac{\pi}{2}\right)$. Similarly for the other two quadrants. ✓

(ii) RTP $\tan\left\{(2n+1)\frac{\pi}{4}\right\} = (-1)^n$

Step 1. At $n=1$

$LHS = \tan\frac{3\pi}{4}$

$= -1.$

$RHS = (-1)^1$

$= LHS$

$\therefore$ Hypothesis is true for $n=1$. ✓

Step 2. Assume true for k.

$\tan\left\{(2k+1)\frac{\pi}{4}\right\} = (-1)^k$ for some k

Step 3. For $n=k+1$:

$$\tan\left\{(2n+1)\frac{\pi}{4}\right\} = \tan\left\{\left[2(k+1)+1\right]\frac{\pi}{4}\right\}$$

6 con'd

$$= \tan\left\{(2k+3)\frac{\pi}{4}\right\}$$

$$= \tan\left\{(2k+1)\frac{\pi}{4} + \frac{\pi}{2}\right\}$$

$$= -\cot\left[(2k+1)\frac{\pi}{4}\right] \quad (\text{from ii}) \quad ✓$$

$$= -\frac{1}{(-1)^k} \quad ✓$$

$$= (-1)^{k+1}$$

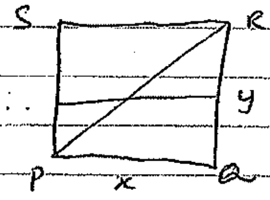
$\Rightarrow$ If hypothesis is true for $n=k$, then it is true for $n=k+1$.

Step 4: Since if the hypothesis is true for $n=k$ it is true for $n=k+1$, and since it is true for $n=1$ it is true for $n=1+1=2$. Since true for $n=2$ it is true for $n=2+1=3$ and so on for all integer $n \geq 1$ ✓ ④

7 (a) Call $PQ = x$ ✓

$RQ = y$

$\therefore \quad y = \frac{A}{x}$ ✓



Total length of red tape

$$\mathcal{L} = x + \sqrt{x^2+y^2}$$

$$= x + \sqrt{x^2 + \left(\frac{A}{x}\right)^2} \quad ✓$$

$$\mathcal{L}' = 1 + \frac{1}{2}\left(x^2+\left(\frac{A}{x}\right)^2\right)^{-\frac{1}{2}} \cdot \left(2x - 2\frac{A^2}{x^3}\right) \quad ✓$$

$$= 0 \quad \text{for TP :}$$

$$\frac{1}{2}\left(x^2+\left(\frac{A}{x}\right)^2\right)^{-\frac{1}{2}}\left(2x - \frac{A^2}{x^3}\right) = -1$$

$$\frac{2A^2}{x^3} - 2x = 2\sqrt{\left(x^2+\left(\frac{A}{x}\right)^2\right)}$$

$$\left(\frac{2A^2}{x^3} - 2x\right)^2 = 4\left(x^2+\left(\frac{A}{x}\right)^2\right)$$

$$\therefore \quad \frac{4A^4}{x^6} - 2.2.\frac{A^2}{x^3}.2x + 4x^2 = 4x^2 + \frac{4A^2}{x^2}$$

$$\frac{A^2}{x^6} - \frac{2}{x^2} = \frac{1}{x^2}$$

$$\frac{A^2}{x^6} = \frac{3}{x^2}.$$

$$\frac{A^2}{3} = x^4$$

$$x = \frac{A^{1/2}}{3^{1/4}} \quad ✓$$

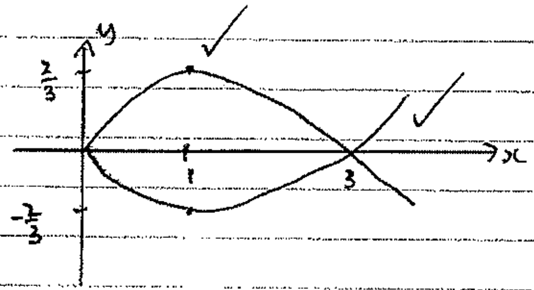
(As only value +ve assuming only value of gives min)
(it is but ignoring f'' or change of sign)

Hence Dimension of notice board are :

$$\frac{A^{\frac{1}{2}}}{3^{1/4}} \quad \text{and} \quad \frac{A}{\frac{A^{1/2}}{3^{1/4}}} = A^{\frac{1}{2}}.3^{1/4}.$$

✓ ✓ ⑦

(b) (i)



④ - Shape
- T.P.
- y'
- y

$$y = \pm \frac{1}{\sqrt{9}}\sqrt{x}(x-3) = \pm \frac{1}{3}(x^{3/2} - 3x^{\frac{1}{2}}) \quad ✓$$

$$y' = \pm\left(\frac{x^{1/2}}{2} - \frac{1}{2}x^{-\frac{1}{2}}\right) = 0 \quad \text{at } x = 1. \quad ✓$$

(ii)

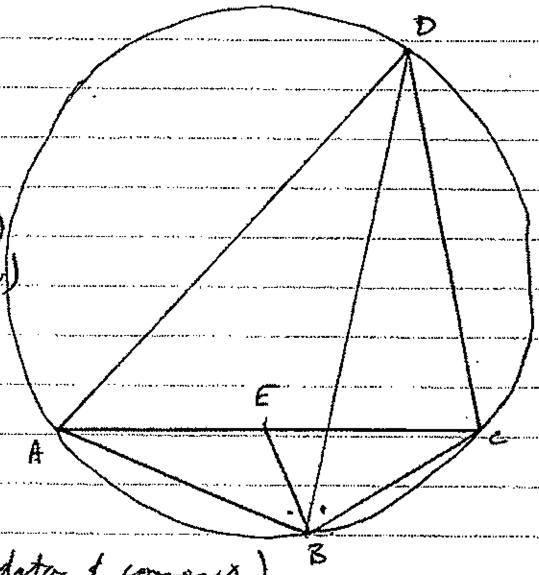
$$A = -\int_0^3 \frac{1}{3}(x^{3/2} - 3x^{\frac{1}{2}}).dx \quad ✓$$

$$= -\left[\frac{1}{3}.\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right]_0^3 \quad ✓ \quad ④$$

$$= -\frac{1}{3}.\frac{2}{5}.3^{5/2} + \frac{2}{3}.3^{3/2} \quad ✓$$

$$= \frac{2}{3}.3.\sqrt{3} - \frac{1}{3}.\frac{2}{5}.9\sqrt{3} = \frac{10\sqrt{3} - 6\sqrt{3}}{5} = \frac{4\sqrt{3}}{5}. \quad ✓$$

8 (a) (i)



In $\triangle ABE$ & $\triangle DBC$

✓ $\angle ABE = \angle DBC$ (data)

✓ $\angle BAE = \angle BDC$ (standg

$\angle AEB = \angle DCB$ ($\angle$ sum Δ) on same arc)

✓ $\therefore \triangle ABE \,|||\, \triangle DBC$ (equiangular)

In $\triangle ABD$ & $\triangle EBC$

✓ $\angle ABD = \angle ABE + \angle EBD$

$= \angle DBC + \angle EBD$

$= \angle EBC$. (data & common $\angle$)

✓ $\angle ECB = \angle ADB$ ($\angle$'s standing on same arc AB)

$\therefore \angle DAB = \angle BEC$ ($\angle$ sum Δ)

✓ $\therefore \triangle ABD \,|||\, \triangle EBC$ (equiangular) ⑥

(ii) $\dfrac{AB}{DB} = \dfrac{BE}{BC} = \dfrac{AE}{DC}$ by similar Δ's.

Also $\dfrac{AB}{EB} = \dfrac{BD}{BC} = \dfrac{AD}{EC}$ ✓

Hence $AE = \dfrac{AB}{DB} \cdot DC$

$EC = AD \cdot \dfrac{BC}{BD}$ } ✓ (either)

And $AE + EC = AC$ ✓

$\therefore \dfrac{AB}{DB} \cdot DC + \dfrac{AD \cdot BC}{BD} = AC$ ✓ ④

$\therefore AB \cdot DC + AD \cdot BC = AC \cdot DB$.

8 con'd.

(b) $AB . DC + AD . BC = AC . DB$ (in (ii)) ✓

$\Rightarrow AP . CB + AC . PB = AB . CP$ ✓

$\therefore PC = \dfrac{AP . CB}{AB} + \dfrac{AC . PB}{AB}$ ✓

$= AP . \dfrac{CB}{CB} + PB . \dfrac{CB}{CB}$ (△ABC equilat) ✓

$= AP + PB .$ ✓ ⑤